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All the special functions are fractional differintegrals of elementary functions

Virginia Kiryakova†
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1090 Sofia, Bulgaria
Instituto per la Ricerca di Base, Monteroduni, Italy

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Abstract. In this survey we discuss a unified approach to the generalized hypergeometric functions based on a *generalized fractional calculus* developed in the monography by Kiryakova. This generalization of the classical theory of the operators of integration and differentiation of fractional order deals with integral (differintegral) operators involving *Meijer's G-* and *Fox's H-functions* as kernel functions. Their theory is fully developed and illustrated by various special cases and applications in different areas of the applicable analysis.

Usually, the special functions of mathematical physics are defined by means of power series representations. However, some alternative representations can be used as their definitions. Let us mention the well known *Poisson integrals* for the Bessel functions and the analytical continuation of the Gauss hypergeometric function via the *Euler integral formula*. The *Rodrigues differential formulae*, involving repeated or fractional differentiation are also used as definitions of the classical orthogonal polynomials and their generalizations. As to the other special functions (most of them being ${}_pF_q$- and ${}_p\Psi_q$-functions), such representations are less popular and even unknown in the general case. There exist various integral and differential formulae, but, unfortunately, they are quite peculiar for each corresponding special function and scattered in the literature without any common idea to relate them. Here, all the *generalized hypergeometric functions* are proved to be generalized fractional integrals or derivatives of three basic elementary functions. On this base, they are classified in three specific classes and several new integral and differential representations are found.

1. Introduction

The generalized fractional calculus developed in [11] is based on the notion of *generalized operators of fractional integration* of Riemann–Liouville type

$$If(x) = x^{\delta}\int_0^1 \Phi(\sigma)\sigma^{\gamma} f(x\sigma)\,\mathrm{d}\sigma \tag{1.1}$$

(see Kalla [6]), where $\Phi(\sigma)$ is an arbitrary elementary or special kernel function. However, in order to develop a meaningful detailed theory with practical applications, we choose the kernel functions as suitable special cases of the *Meijer's G-functions* and *Fox's H-functions*.

Definition 1.1. (see [14, 18]) By a *Fox's H-function* we mean the generalized hypergeometric function defined by means of the contour integral

$$H^{m,n}_{p,q}(\sigma) = H^{m,n}_{p,q}\left[\sigma\,\middle|\,\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}\right]$$

† E-mail address: virginia@math.acad.bg, virginia@bgearn.acad.bg

$$= H^{m,n}_{p,q}\left[\sigma\left|\begin{matrix}(a_j, A_j)_1^p\\ (b_k, B_k)_1^q\end{matrix}\right.\right] = \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s)\sigma^s\,\mathrm{d}s \qquad \sigma \neq 0 \tag{1.2}$$

where the integrand in (1.2) has the form

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod_{k=1}^m \Gamma(b_k - B_k s)\prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + B_k s)\prod_{j=n+1}^p \Gamma(a_j - A_j s)}$$

and $\mathcal{L}$ is a suitable contour in $\mathbb{C}$; the orders $(m, n;\, p, q)$ are non-negative integers such that $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant q$; the parameters A_j, $j = 1, \dots, p$ and B_k, $k = 1, \dots, q$ are positive and a_j, $j = 1, \dots, p$, b_k, $k = 1, \dots, q$, are arbitrary complex numbers such that

$$A_j(b_k + l) \neq B_k(a_j - l' - 1) \qquad l, l' = 0, 1, 2, \dots,\ j = 1, \dots, p, k = 1, \dots, q.$$

In particular, when all $A_j = B_k = 1$, we obtain the so-called *Meijer's G-function* [5, vol 1]),

$$H^{m,n}_{p,q}\left[\sigma\left|\begin{matrix}(a_j, 1)_1^p\\ (b_k, 1)_1^q\end{matrix}\right.\right] = G^{m,n}_{p,q}\left[\sigma\left|\begin{matrix}(a_j)_1^p\\ (b_k)_1^q\end{matrix}\right.\right] \tag{1.3}$$

namely,

$$G^{m,n}_{p,q}\left[\sigma\left|\begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_q\end{matrix}\right.\right] = \frac{1}{2\pi \mathrm{i}}\int_{\mathcal{L}} \frac{\prod_{k=1}^m \Gamma(b_k - s)\prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + s)\prod_{j=n+1}^p \Gamma(a_j - s)}\sigma^s\,\mathrm{d}s. \tag{1.4}$$

In section 2 we define our generalized fractional integrals and derivatives using as kernel functions peculiar cases of the above special functions with $m = p = q$, $n = 0$. This choice of the kernel function $\Phi(\sigma)$ ensures a decomposition of these operators (called also *multiple Erdélyi–Kober operators*) into products of commuting classical Erdélyi–Kober (E–K) operators. Thus, complicated multiple integrals or differintegral expressions can be represented alternatively by means of single integrals involving special functions. The beauty and succinctness of the notation and properties of these functions allow the development of a full chain of operational rules, mapping properties and convolutional structure of the generalized fractional integrals as well as an appropriate explicit definition of the corresponding generalized derivatives. On the other hand, the frequent appearance of compositions of classical Riemann–Liouville and Erdélyi–Kober fractional operators in various problems of applied analysis gives the key to the great number of applications and known special cases of our generalized fractional differintegrals.

Section 3 deals with the generalized hypergeometric functions ${}_pF_q(x)$ being also special cases of the Meijer's G-functions (see [5, 14]):

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} G^{1,p}_{p,q+1}\left[-x\left|\begin{matrix}1 - a_1, \dots, 1 - a_p\\ 0, 1 - b_1, \dots, 1 - b_q\end{matrix}\right.\right]. \tag{1.5}$$

Definition 1.2. By a *generalized hypergeometric function (GHF)* ${}_pF_q(x)$ we mean the sum of the GHF series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) = {}_pF_q((a_i)_1^p; (b_j)_1^q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \dots (a_p)_k}{(b_1)_k(b_2)_k \dots (b_q)_k} \cdot \frac{x^k}{k!}$$

$$\text{where } (a)_0 = 1,\ (a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} \tag{1.6}$$

in the domain of its convergence: $\Omega = \{|x| < \infty\}$ for $p \leqslant q$ and $\Omega = \{|x| < 1\}$ for $p = q + 1$, or its analytical continuation in $\{|x| > 1, |\arg(1 - x)| < \pi\}$ in the latter case. One may consider x also as a real variable $x \in [0, \infty)$.

We separate the ${}_pF_q$-functions into three classes depending on whether $p < q$, $p = q$ or $p = q + 1$ and represent the functions of each class as generalized fractional integrals or derivatives of three basic elementary functions:

$$\cos_{q-p+1}(x) \text{ (if } p < q) \qquad x^\alpha \exp x \text{ (if } p = q) \qquad x^\alpha (1-x)^\beta \text{ (if } p = q+1). \tag{1.7}$$

The above-mentioned representations lead to several new integral and differential formulae for the ${}_pF_q$-functions and allow their study in a unified way. Many interesting particular cases are mentioned.

In section 4 we continue the same approach to the so-called Wright's generalized hypergeometric functions ([18, 19])

$$\begin{aligned} {}_p\Psi_q\left[\begin{matrix}(a_1, A_1), \ldots, (a_p, A_p)\\ (b_1, B_1), \ldots, (b_q, B_q)\end{matrix}; x\right] &= \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + kA_1)\ldots\Gamma(a_p + kA_p)}{\Gamma(b_1 + kB_1)\ldots\Gamma(b_q + kB_q)} \frac{x^k}{k!} \\ &= H^{1,p}_{p,q+1}\left[-x \left| \begin{matrix}(1-a_1, A_1), \ldots, (1-a_p, A_p)\\ (0, 1), (1-b_1, B_1), \ldots, (1-b_q, B_q)\end{matrix}\right.\right]. \end{aligned} \tag{1.8}$$

Naturally,

$${}_p\Psi_q\left[\begin{matrix}(a_1, 1), \ldots, (a_p, 1)\\ (b_1, 1), \ldots, (b_q, 1)\end{matrix}; x\right] = \frac{\prod_{i=1}^{q}\Gamma(b_i)}{\prod_{j=1}^{p}\Gamma(a_j)}\, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x). \tag{1.9}$$

The results for the special functions (1.9) are essentially new and are published for the first time.

2. Generalized fractional calculus

We introduce the following generalizations of the Riemann–Liouville (R–L) fractional integrals of order $\delta > 0$:

$$R^\delta f(x) = \frac{1}{\Gamma(\delta)} \int_0^x (x-\tau)^{\delta-1} f(\tau)\,\mathrm{d}\tau = \frac{x^\delta}{\Gamma(\delta)} \int_0^1 (1-\sigma)^{\delta-1} f(x\sigma)\,\mathrm{d}\sigma \tag{2.1}$$

having the form of operators (1.1).

Definition 2.1. Let $m \geqslant 1$ be integer, $\beta > 0$, $\gamma_1, \ldots, \gamma_m$ and $\delta_1 \geqslant 0, \ldots, \delta_m \geqslant 0$ be arbitrary real numbers. By a generalized (multiple, m-tuple) E–K operator of integration of multiorder $\delta = (\delta_1, \ldots, \delta_m)$ we mean an integral operator

$$I^{(\gamma_k),(\delta_k)}_{\beta,m} f(x) = \int_0^1 G^{m,0}_{m,m}\left[\sigma \left| \begin{matrix}(\gamma_k + \delta_k)_1^m\\ (\delta_k)_1^m\end{matrix}\right.\right] f(x\sigma^{1/\beta})\,\mathrm{d}\sigma. \tag{2.2}$$

Then, each operator of the form

$$Rf(x) = x^{\beta\delta_0} I^{(\gamma_k),(\delta_k)}_{\beta,m} f(x) \qquad \text{with arbitrary } \delta_0 \geqslant 0 \tag{2.3}$$

is said to be a *generalized (m-tuple) operator of fraction integration of R–L type*, or briefly a *generalized R–L fractional integral*.

Generalizing further the operators of fractional calculus, in [8, 11] we consider also operators involving Fox's H-functions instead of the G-functions in (2.2) and (2.3). They are named in the same way, namely *generalized (multiple) E–K operators (fractional integrals)*:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) = \begin{cases} \displaystyle\int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix}(\gamma_k + \delta_k + 1 - (1/\beta_k), (1/\beta_k))_1^m\\ (\gamma_k + 1 - (1/\beta_k), (1/\beta_k))_1^m\end{matrix}\right.\right] f(x\sigma)\,\mathrm{d}\sigma \\ \qquad\qquad \text{if } \displaystyle\sum_{k=1}^m \delta_k > 0 \\ f(x) \qquad\quad \text{if } \delta_1 = \delta_2 = \cdots = \delta_m = 0. \end{cases} \tag{2.4}$$

Thus, along with the multiorder of integration $(\delta_1, \ldots, \delta_m)$ and the multiweight $(\gamma_1, \ldots, \gamma_m)$, we introduce also a multiparameter $(\beta_1 > 0, \ldots, \beta_m > 0)$ (different β_k's) instead of the same $\beta > 0$ in the case with kernel G-function. Note that due to the relation generalizing (1.3),

$$H^{m,n}_{p,q}\left[\sigma \left| \begin{matrix} (a_1, 1/\beta), \ldots, (a_p, 1/\beta) \\ (b_1, 1/\beta), \ldots, (b_q, 1/\beta) \end{matrix} \right.\right] = \beta G^{m,n}_{p,q}\left[\sigma^\beta \left| \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix} \right.\right] \qquad \beta > 0 \quad (2.5)$$

operator (2.4) involving a H-function reduces to its simpler form (2.2), viz

$$\text{for } \beta_1 = \beta_2 = \cdots = \beta_m = \beta > 0 \qquad I^{(\gamma_k),(\delta_k)}_{(\beta,\beta,\ldots,\beta),m} = I^{(\gamma_k),(\delta_k)}_{\beta,m}. \quad (2.6)$$

Now let us define generalizations of the classical R–L derivatives of fractional order $\delta > 0$:

$$D^\delta f(x) = \begin{cases} \dfrac{\mathrm{d}^\delta}{\mathrm{d}x^\delta} f(x) = f^{(\delta)}(x) & \text{for integer } \delta, \\ \dfrac{\mathrm{d}^\eta}{\mathrm{d}x^\eta} R^{\eta-\delta} f(x) & \text{for non-integer } \delta \text{ with } \eta = [\delta] + 1 \end{cases} \quad (2.7)$$

corresponding to generalized fractional integrals (2.2) and (2.4).

Definition 2.2. With the same parameters as in definition 2.1 and the integers

$$\eta_k = \begin{cases} \delta_k & \text{if } \delta_k \text{ is integer} \\ [\delta_k] + 1 & \text{if } \delta_k \text{ is non-integer} \end{cases} \qquad k = 1, \ldots, m \quad (2.8)$$

we introduce the auxiliary differential operator

$$D_\eta = \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r} x \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_r + j\right)\right]. \quad (2.9)$$

Then, the multiple (m-tuple) E–K fractional derivatives of multiorder $\delta = (\delta_1 \geqslant 0, \ldots, \delta_m \geqslant 0)$ are defined by means of the differintegral operators:

$$\begin{aligned} D^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) &= D_\eta I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{(\beta_k),m} f(x) \\ &= D_\eta \int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_k + \eta_k + 1 - (1/\beta_k), (1/\beta_k))_1^m \\ (\gamma_k + 1 - (1/\beta_k), (1/\beta_k))_1^m \end{matrix} \right.\right] f(x\sigma)\,\mathrm{d}\sigma. \end{aligned} \quad (2.10)$$

In the case (2.5) of equal β_k's we obtain simpler representations involving the Meijer's G-function and corresponding to generalized fractional integrals (2.2):

$$D^{(\gamma_k),(\delta_k)}_{\beta,m} = D_\eta I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{\beta,m} = \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta} x \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_r + j\right)\right] I^{(\gamma_k+\delta_k),(\eta_k-\delta_k)}_{\beta,m}. \quad (2.11)$$

More generally, the differintegral operators of the form

$$Df(x) = D^{(\gamma_k),(\delta_k)}_{\beta,m} x^{-\delta_0} f(x) = x^{-\delta_0} D^{(\gamma_k-(\delta_0/\beta)),(\delta_k)}_{\beta,m} f(x) \qquad \text{with } \delta_0 \geqslant 0 \quad (2.12)$$

are called generalized (multiple, m-tuple) fractional derivatives.

Generalized derivatives (2.11) and (2.12) are the counterparts of the generalized fractional integrals (2.2) and (2.3).

The generalized fractional integrals and derivatives include as special cases a great number of operators of fractional (or integer but generalized) integration and differentiation, even in the simpler cases related to the Meijer's G-function. We consider separately the cases $m = 1, 2$ (when many generalized integration and differentiation operators introduced and used by various authors are included as special cases) and $m > 2$ (when our operators are less known).

(i) *m = 1.* The kernel function of (2.2) is the elementary function

$$G^{1,0}_{1,1}\left[\sigma\left|\begin{matrix}\gamma+\delta\\ \gamma\end{matrix}\right.\right]=\begin{cases}(1-\sigma)^{\delta-1}\sigma^{\gamma}/\Gamma(\delta) & 0<\sigma<1\\ 0, & \sigma>1.\end{cases}\tag{2.13}$$

Thus, for arbitrary $\beta > 0$, γ and $\delta > 0$ the generalized fractional integrals (2.2) coincide with the well known E–K operators (integrals)

$$I^{\gamma,\delta}_{\beta}f(x)=\int_0^1\frac{(1-\sigma)^{\delta-1}\sigma^{\gamma}}{\Gamma(\delta)}f(x\sigma^{1/\beta})\,\mathrm{d}\sigma=I^{\gamma,\delta}_{\beta,1}f(x)\tag{2.14}$$

widely used in the applied analysis (see e.g. [17]) and incorporating the R–L fractional integrals (2.1) as well: $R^{\delta}f(x)=x^{\delta}I^{0,\delta}_{1}f(x)$.

For $m = 1$ the generalized fractional derivative (2.10), corresponding to the E–K fractional integral (2.14), is called in [11] in E–K fractional derivative and has the representation

$$\begin{aligned}D^{\gamma,\delta}_{\beta}f(x)&:=D^{\gamma,\delta}_{\beta,1}f(x)=D_{\eta}I^{\gamma+\delta,\eta-\delta}_{\beta}f(x)\\ &=\left[\prod_{j=1}^{\eta}\left(\frac{1}{\beta}x\frac{\mathrm{d}}{\mathrm{d}x}+\gamma+j\right)\right]\int_0^1\frac{(1-\sigma)^{\eta-\delta-1}\sigma^{\gamma+\delta}}{\Gamma(\eta-\delta)}f(x\sigma^{1/\beta})\,\mathrm{d}\sigma.\end{aligned}\tag{2.15}$$

Symbolically, it can be written as

$$D^{\gamma,\delta}_{\beta}f(x):=D^{\gamma,\delta}_{\beta,1}f(x)=[x^{-\gamma}D^{\delta}x^{\gamma+\delta}f(x^{1/\beta})]_{x\to x^{\beta}}\tag{2.15$'$}$$

where D^{δ} is the R–L fractional derivative (2.7), being also an E–K derivative:

$$D^{\delta}=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\delta}=x^{-\delta}D^{-\delta,\delta}_{1}=D^{0,\delta}_{1}x^{-\delta}\qquad \delta>0.$$

(ii) *m = 2.* Then, the kernel function of (2.2) is the Gauss hypergeometric function ${}_2F_1$, namely

$$\begin{aligned}&G^{2,0}_{2,2}\left[\sigma\left|\begin{matrix}\gamma_1+\delta_1,\gamma_2+\delta_2\\ \gamma_2,\gamma_2\end{matrix}\right.\right]\\ &\quad=\begin{cases}\dfrac{\sigma^{\gamma_2}(1-\sigma)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)}\,{}_2F_1(\gamma_2+\delta_2-\gamma_1,\delta_1;\delta_1+\delta_2;1-\sigma) & \text{for } \sigma<1\\ 0 & \text{for } \sigma>1\end{cases}\end{aligned}\tag{2.16}$$

and the operators $I^{(\gamma_k),(\delta_k)}_{\beta,2}$ are the so-called hypergeometric fractional integrals:

$$\begin{aligned}Hf(x)=I^{(\gamma_1,\gamma_2),(\delta_1,\delta_2)}_{\beta,2}f(x)&=\int_0^1\frac{\sigma^{\gamma_2}(1-\sigma)^{\delta_1+\delta_2-1}}{\Gamma(\delta_1+\delta_2)}\,{}_2F_1(\gamma_2+\delta_2-\gamma_1,\delta_1;\delta_1+\delta_2;1-\sigma)\\ &\quad\times f(x\sigma^{1/\beta})\,\mathrm{d}\sigma\end{aligned}\tag{2.17}$$

introduced first by Love [13] and considered in different modifications by Saigo [16].

(iii) *m > 2.* In this case the generalized fractional integrals and derivatives have been used mainly in their alternative, multiple integral representations, without involving special kernel functions. This has caused a lack of suitable tools to deal with them easily.

In a series of papers (e.g. [3]) Dimovski introduced and studied the hyper-Bessel differential operators of arbitrary integer order $m > 1$ of the form

$$B = x^{\alpha_0}\frac{\mathrm{d}}{\mathrm{d}x}x^{\alpha_1}\cdots\frac{\mathrm{d}}{\mathrm{d}x}x^{\alpha_m} = x^{-\beta}\prod_{k=1}^{m}\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\beta\gamma_k\right) \qquad \beta > 0. \tag{2.18}$$

These operators, generalizing the second-order Bessel differential operator are widely used in the differential equations of mathematical physics (see [11, ch 3]). Dimovski developed a detailed theory, including operational calculi, integral transforms, etc. Operators (2.18) and their linear right inverse operators L, the so-called hyper-Bessel integral operators have been shown in [11] to be generalized (m-tuple, arbitrary $m > 1$) fractional derivatives and integrals of integer multi-order ($\delta_1 = 1, \ldots, \delta_m = 1$), namely

$$L = \frac{x^{\beta}}{\beta^m} I^{(\gamma_k),(1)}_{\beta,m} f(x) \qquad B = \beta^m D^{(\gamma_k),(1)}_{\beta,m} x^{-\beta} = \frac{\beta^m}{x^{\beta}} D^{(\gamma_k-1),(1)}_{\beta,m}. \tag{2.19}$$

A variety of useful transmutation operators, related to operators (2.18)–(2.19), such as the Poisson–Sonine–Dimovski transforms are also examples of m-tuple generalized fractional integrals, for all the details see ch 3 of [11].

The multiple Dzrbashjan–Gelfond–Leontiev operators ([11, section 5.4]), are typical examples of the more complicated generalized fractional (differ) integrals, (2.4) and (2.10) involving Fox's H-functions.

In [11] the generalized operators of fractional integration and differentiation have been considered in different *functional spaces*, such as weighted spaces of continuous, Lebesgue integrable or analytic functions. Here we need the definition of the latter spaces only.

Definition 2.3. Let μ be arbitrary real, the variable x be real or complex, running respectively over the interval $[0,\infty)$ or in the domain $\Omega \subset \mathbb{C}$, starlike with respect to the origin $x = 0$, and let $\mathcal{H}(\Omega)$ stand for the space of analytic functions in Ω. Denote

$$\mathcal{H}_\mu(\Omega) = \{f(x) = x^{\mu}\tilde{f}(x);\ \tilde{f}(x) \in \mathcal{H}(\Omega)\} \qquad \mathcal{H}_0(\Omega) := \mathcal{H}(\Omega). \tag{2.20}$$

To study the generalized fractional integrals, we use essentially the theory of the G- and H-functions, appearing as kernel functions of (2.2) and (2.4). To this end we refer to the classical book [5, ch 5], and also to [14, 18, 19] or [11, appendix]. Note also that the $G^{m,0}_{m,m}$- and $H^{m,0}_{mm}$-functions have three regular singular points $\sigma = 0, 1$ and ∞, they vanish for $|\sigma| > 1$ and are analytic functions in the unit disk $|\sigma| < 1$. Their asymptotic behaviour near $\sigma = 0, 1$ is already well known (see e.g. [14] or [11, appendix]) and ensures the correctness of definitions (2.2) and (2.4) in the above spaces under suitable conditions on the parameters.

Operators (2.4) can be rewritten in the form

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) = \frac{1}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\,\middle|\,\begin{matrix}(\gamma_k+\delta_k+1-(1/\beta_k),(1/\beta_k))_1^m\\(\gamma_k+1-(1/\beta_k),(1/\beta_k))_1^m\end{matrix}\right] f(t)\,\mathrm{d}t$$

and thus this can be put in the form of a convolutional-type integral transform,

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) = \int_0^\infty k\left(\frac{x}{t}\right) f(t)\frac{\mathrm{d}t}{t} = (k \circ f)(x)$$

where $\circ$ denotes the Mellin convolution.

Most of the basic results for the operators of the generalized fractional calculus have been stated in [11] separately for the cases of G- and H-functions and for different kinds of functional spaces. Here we expose them in the general case (2.4) only. The corresponding reductions for the simpler case (2.2) with G-functions are easily seen (for proofs and details see also [9]).

Theorem 2.4. Let the conditions

$$\gamma_k > -\frac{\mu}{\beta_k} - 1 \qquad \delta_k > 0 \qquad k = 1, \ldots, m \tag{2.21}$$

be satisfied. Then, the multiple E–K operator (2.4) maps the class $\mathcal{H}_\mu(\Omega)$ into itself, preserving the power functions up to constant multipliers:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{x^p\} = c_p x^p,\ p > \alpha \qquad \text{where } c_p = \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + (p/\beta_k) + 1)}{\Gamma(\gamma_k + \delta_k + (p/\beta_k) + 1)}. \tag{2.22}$$

The image of a power series

$$f(x) = x^\mu \sum_{n=0}^{\infty} a_n x^n = x^\mu(a_0 + a_1 x + \cdots) \in \mathcal{H}_\mu(\Delta_R) \qquad \Delta_R = \{|x| < R\}$$

where $R = \{\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}\}^{-1}$, is given by the series

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) = x^\mu \sum_{n=0}^{\infty} \left\{ a_n \prod_{k=1}^{m} \frac{\Gamma(\gamma_k + (n+\mu)/\beta_k + 1)}{\Gamma(\gamma_k + \delta_k + (n+\mu)/\beta_k + 1)} \right\} x^n \tag{2.23}$$

having the same radius of convergence $R > 0$ and the same signs of the coefficients.

From the properties of the G- and H-functions some immediate corollaries of definitions (2.2) and (2.4) follow.

Theorem 2.5. Suppose conditions (2.21) are satisfied. Then, in $\mathcal{H}_\mu(\Omega)$ the following basic operational rules of multiple E–K fractional integrals (2.4) hold:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{\lambda f(cx) + \eta g(cx)\} = \lambda\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f\}(cx) + \eta\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} g\}(cx) \tag{2.24}$$

(bilinearity);

$$I^{(\gamma_1,\ldots,\gamma_s,\gamma_{s+1},\ldots,\gamma_m),(0,\ldots,0,\delta_{s+1},\ldots,\delta_m)}_{(\beta_1,\ldots,\beta_m),m} f(x) = I^{(\gamma_{s+1},\ldots,\gamma_m)(\delta_{s+1},\ldots,\delta_m)}_{(\beta_{s+1},\ldots,\beta_m),m-s} f(x) \tag{2.25}$$

(if $\delta_1 = \delta_2 = \cdots = \delta_s = 0$, then the multiplicity reduces to $(m-s)$);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} x^\lambda f(x) = x^\lambda I^{(\gamma_k+(\lambda/\beta_k)),(\delta_k)}_{(\beta_k),m} f(x),\ \lambda \in \mathbb{R} \tag{2.26}$$

(generalized commutability with power functions);

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} f(x) = I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) \tag{2.27}$$

(commutability of operators of form (2.4));

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} I^{(\tau_j),(\alpha_j)}_{(\varepsilon_j),n} f(x) = I^{((\gamma_k)_1^m,(\tau_j)_1^n)((\delta_k)_1^m,(\alpha_j)_1^n)}_{((\beta_k)_1^m,(\varepsilon_j)_1^n),m+n} f(x) \tag{2.28}$$

(compositions of m-tuple and n-tuple integrals (2.4) are $(m+n)$-tuple integrals);

$$I^{(\gamma_k+\delta_k),(\sigma_k)}_{(\beta_k),m} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) = I^{(\gamma_k),(\sigma_k+\delta_k)}_{(\beta_k),m} f(x) \qquad \text{if } \delta_k > 0, \sigma_k > 0, k = 1, \ldots, m \tag{2.29}$$

(law of indices, product rule or semigroup property);

$$\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\}^{-1} f(x) = I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),m} f(x) \tag{2.30}$$

formal inversion formula).

The above inversion formula follows from index law (2.29) for $\sigma_k = -\delta_k < 0$, $k = 1, \ldots, m$ and definition (2.4) for the zero multiorder of integration, since

$$I_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = I_{(\beta_k),m}^{(\gamma_k),(0,\ldots,0)} f(x) = f(x).$$

However , symbols (2.4) have not yet been defined for negative multiorders of integration $-\delta_k < 0$, $k = 1, \ldots, m$. The problem is to propose an appropriate meaning for them and hence to avoid the divergent integrals appearing in (2.30). The situation is the same as in the classical case when the R–L and E–K operators of fractional order $\delta > 0$ are inverted by appealing to an additional differentiation of suitable integer order $\eta = [\delta] + 1$. Now, we make use of the following differential formula for the kernel H-function ([11], lemma 5.1.7 or lemma B.3, appendix, for the G-function). Let $\eta_k \geqslant 0$, $k = 1, \ldots, m$ be arbitrary integers, then

$$H_{m,m}^{m,0}\left[\frac{t}{x}\left|\begin{matrix}(a_k, 1/\beta_k)_1^m\\(b_k, 1/\beta_k)_1^m\end{matrix}\right.\right] = D_\eta H_{m,m}^{m,0}\left[\frac{t}{x}\left|\begin{matrix}(a_k+\eta_k, 1/\beta_k)_1^m\\(b_k, 1/\beta_k)_1^m\end{matrix}\right.\right] \tag{2.31}$$

with the differential operator D_η being a polynomial of $x\,\mathrm{d}/\mathrm{d}x$ of degree $\eta = \eta_1 + \cdots + \eta_m$:

$$D_\eta = \prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r}x\frac{\mathrm{d}}{\mathrm{d}x} + a_r - 1 + j\right).$$

This formula helps to increase the parameters a_k, $k = 1, \ldots, m$, of the H-function in the upper row by arbitrary integers $\eta_k \geqslant 0$, $k = 1, \ldots, m$, by using a suitable operator D_η. Choosing appropriately the necessary parameters, as in definition 2.2, we can prove now that $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ (2.10) is in fact a generalized fractional derivative with a linear right inverse operator $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$, namely

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) = f(x) \qquad f \in \mathcal{H}_\mu(\Omega). \tag{2.32}$$

Now we state the basic result for the generalized fractional integrals (2.2) and (2.4) *suggesting their alternative name 'multiple (m-tuple)' fractional integrals*.

Theorem 2.6. (The composition/decomposition theorem.) Under the conditions (2.21), the classical E–K fractional integrals of the form (2.14), $I_{\beta_k}^{\gamma_k,\delta_k}$, $k = 1, \ldots, m$, commute in $\mathcal{H}_\mu(\Omega)$ and their product

$$\begin{aligned} I_{\beta_m}^{\gamma_m,\delta_m}\{I_{\beta_{m-1}}^{\gamma_{m-1},\delta_{m-1}} \ldots (I_{\beta_1}^{\gamma_1,\delta_1} f(x))\} &= \left[\prod_{k=1}^{m} I_{\beta_k}^{\gamma_k,\delta_k}\right] f(x) \\ &= \int_0^1 \underbrace{\ldots}_{m} \int_0^1 \left[\prod_{k=1}^{m}\frac{(1-\sigma_k)^{\delta_k-1}\sigma_k^{\gamma_k}}{\Gamma(\delta_k)}\right] f(x\sigma_1^{1/\beta_1}\ldots\sigma_m^{1/\beta_m})\,\mathrm{d}\sigma_1\ldots\mathrm{d}\sigma_m \end{aligned} \tag{2.33}$$

can be represented as an m-tuple E–K operator (2.4), i.e. by means of a single integral involving the H-function:

$$\begin{aligned} \left[\prod_{k=1}^{m} I_{\beta_k}^{\gamma_k,\delta_k}\right] f(x) &= I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(x) \\ &= \int_0^1 H_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}(\gamma_k+\delta_k+1-(1/\beta_k), (1/\beta_k))_1^m\\(\gamma_k+1-(1/\beta_k), (1/\beta_k))_1^m\end{matrix}\right.\right] f(x\sigma)\,\mathrm{d}\sigma \qquad f \in \mathcal{H}_\mu. \end{aligned} \tag{2.34}$$

Conversely, under the same conditions, each multiple E–K operator of form (2.4) can be represented as a product (2.33).

Let us note that the same proposition also holds for the generalized fractional derivatives (2.10) and (2.11): they are products of E–K derivatives (2.15), namely:

$$D_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{\beta_1}^{\gamma_1,\delta_1} D_{\beta_2}^{\gamma_2,\delta_2} \dots D_{\beta_m}^{\gamma_m,\delta_m}. \tag{2.35}$$

Combining (2.30), (2.32), (2.34) and (2.35), we can make the next step in clarifying the structure of a great number of operators—generalized or classical, fractional or integer-order integrations, differentiations or differintegrations. Namely, in [11] we have introduced a unified theory based on the common notion 'generalized fractional differintegrals'. By now, operators $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ with all $\delta_k \geqslant 0$, $k = 1, \dots, m$, have been considered as (fractional) integrals while those with all $\delta_k < 0$, $k = 1, \dots, m$, have been undertaken as formal denotations for the generalized fractional derivatives (cf (2.30) and (2.32)): $I_{(\beta_k),m}^{(\gamma_k'+\delta_k'),(-\delta_k')} = D_{(\beta_k),m}^{(\gamma_k'),(\delta_k')}$, i.e. $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} = D_{(\beta_k),m}^{(\gamma_k+\delta_k),(-\delta_k)}$. Now, having the decomposition theorem in mind, we may consider both symbols $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ and $D_{(\beta_k),m}^{(\gamma_k),(\delta_k)}$ as generalized fractional differintegrals. If not all of the components of multiorder of 'differintegration' $\delta = (\delta_1, \dots, \delta_m)$ are of the same sign, we simply interpret them as 'mixed' products of E–K fractional integrals and derivatives. For example, if $\delta_1 < 0, \dots, \delta_s < 0$, $\delta_{s+1} = \dots = \delta_{s+j} = 0$, $\delta_{s+j+1} > 0, \dots, \delta_m > 0$, then

$$\begin{aligned} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} &:= D_{(\beta_1,\dots,\beta_s),s}^{(\gamma_1+\delta_1,\dots,\gamma_s+\delta_s),(-\delta_1,\dots,-\delta_s)} I_{(\beta_{s+j+1},\dots,\beta_m),m-s-j}^{(\gamma_{s+j+1},\dots,\gamma_m),(\delta_{s+j+1},\dots,\delta_m)} \\ &= \prod_{i=1}^{s} D_{\beta_i}^{\gamma_i+\delta_i,-\delta_i} \prod_{k=s+j+1}^{m} I_{\beta_k}^{\gamma_k,\delta_k} \end{aligned} \tag{2.36}$$

is a $(m - j)$-tuple fractional differintegral.

Theorem 2.6 gives the key to the numerous applications of the generalized fractional calculus operators. Some of them, especially those relevant to the theory of the special functions, are briefly mentioned in the next two sections.

3. Representations of the generalized hypergeometric functions ${}_pF_q$

In [10, 11] we have proposed a unified approach to the generalized hypergeometric functions (1.6) and derived new or newly written integral, differential and differintegral representations of these special functions by means of the generalized fractional calculus. A suitable classification of the ${}_pF_q$-functions has also been introduced. The idea is based on the following simple facts: (i) most of the special functions of mathematical physics are nothing but modifications of the GHFs ${}_pF_q$; (ii) each ${}_pF_q$-function can be represented as an E–K fractional differintegral of a ${}_{p-1}F_{q-1}$-function (see (3.1)); (iii) a finite number (q) of steps (ii) leads to one of the basic GHFs ${}_0F_{q-p}$ (for $q - p = 1$, Bessel function), ${}_1F_1$ (confluent HF) and ${}_2F_1$ (Gauss HF); (iv) the above three basic GHFs can be considered as fractional differintegrals of the three elementary functions (1.7), depending on whether $p < q$, $p = q$ or $p = q + 1$; (v) the compositions of the E–K operators arising in (iii) give generalized (q-tuple) fractional integrals or derivatives, according to theorem 2.6. Thus, we obtain the following general result.

Proposition 3.1. All the generalized hypergeometric functions ${}_pF_q$ can be considered as generalized (q-tuple) fractional differintegrals (2.2), (2.3), (2.10) and (2.12) of one of the elementary functions (1.7), depending on whether $p < q$, $p = q$, $p = q + 1$.

To establish the above proposition we need the following basic lemma.

Lemma 3.2. ([1, 12]) Let $|x| < \infty$ ($|x| < 1$ for $p = q + 1$), then

$$[\Gamma(a_p)/\Gamma(b_q)]\,{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \begin{cases} I_{1,1}^{a_p-1, b_q-a_p}\{{}_{p-1}F_{q-1}(a_1, \ldots, a_{p-1}; b_1, \ldots, b_{q-1}; x)\} & \text{if } b_q > a_p \\ D_{1,1}^{b_q-1, a_p-b_q}\{{}_{p-1}F_{q-1}(a_1, \ldots, a_{p-1}; b_1, \ldots, b_{q-1}; x)\} & \text{if } b_q < a_p. \end{cases} \tag{3.1}$$

The three cases $p < q$, $p = q$, $p = q + 1$ are to be considered separately, the first of them being more complicated and involving some auxiliary definitions and results.

First case: $p < q$

Definition 3.3. In [2] Delerue introduced a generalization of the Bessel function $J_\nu(x)$ for a multi-index $\nu = (\nu_1, \ldots, \nu_m)$, $m \geqslant 1$:

$$J_{\nu_1,\ldots,\nu_m}^{(m)}(x) = \frac{(x/m+1)^{\nu_1+\cdots+\nu_m}}{\Gamma(\nu_1+1)\ldots\Gamma(\nu_m+1)}\,{}_0F_m((\nu_k+1)_1^m; -(x/m+1)^{m+1}) \tag{3.2}$$

referred to as a hyper-Bessel function of order m. As a special case, the so-called (generalized) cosine function of order $(m + 1)$ follows,

$$\cos_{m+1}(x) = {}_0F_m\left(\left(\frac{k}{m+1}\right)_1^m; -(x/m+1)^{m+1}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k(m+1)}}{(k(m+1))!} \tag{3.3}$$

generalizing the elementary cosine function $\cos x = \cos_2(x)$, $m = 1$. For more details on (3.2) and (3.3), see [11, appendix; 5, vol 3; 14].

In [4] Dimovski and Kiryakova proved a generalization of the Poisson integral representation of the Bessel function

$$J_\nu(x) = \frac{z}{\sqrt{\pi}}\frac{(x/2)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\cos xt\,\mathrm{d}t \qquad \nu > -\frac{1}{2} \tag{3.4}$$

based on the Poisson–Dimovski transformation (see [11, ch 3]). In terms of the generalized fractional calculus, G- and ${}_pF_q$-functions, the generalized Poisson integral can be written in the modified form ([11], theorem 4.1.1, corollary 4.1.4):

$$\begin{aligned} {}_0F_m((b_k)_1^m; -x) &= c I_{1,m}^{(k/(m+1)-1),(b_k-k/(m+1))}\{\cos_{m+1}((m+1)x^{1/(m+1)})\} \\ &= c\int_0^1 G_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (b_k)_1^m \\ (k/(m+1))_1^m \end{matrix}\right.\right]\sigma^{-1}\cos_{m+1}((m+1)(x\sigma)^{1/(m+1)})\,\mathrm{d}\sigma \end{aligned} \tag{3.5}$$

where $c = \sqrt{(m+1)/(2\pi)^m}\prod_{j=1}^m \Gamma(b_j)$ and the condition $b_k \geqslant k/(m+1)$, $k = 1, \ldots, m$ is supposed. Otherwise, if some of the b_k's does not satisfy this condition, the corresponding component in the generalized fractional differintegral should be considered as an E–K derivative (2.15) (see [11, theorem 4.1.6]). As an illustration, if $b_k := \nu_k + 1 = k/(m+1) - \eta_k$ with integers $\eta_k > 0$, $k = 1, \ldots, m$, then (3.5) turns into a purely differential expression for the 'spherical' hyper-Bessel functions (3.2) ([11, (4.1.42)]), reducible in the case $m = 1$ to the spherical Bessel functions (see e.g. [12])

$$J_{-\eta-\frac{1}{2}}(x) = \frac{(2x)^{\eta+\frac{1}{2}}}{\sqrt{\pi}}\frac{\mathrm{d}\eta}{(\mathrm{d}x^2)^\eta}\left\{\frac{\cos x}{x}\right\} \qquad \eta = 0, 1, 2 \ldots. \tag{3.6}$$

Observe now that by p steps (3.1), a ${}_pF_q$-function, $p < q$ can be reduced to a hyper-Bessel function, i.e. to a ${}_0F_{q-p}$-function:

$${}_pF_q((a_k)_1^p; (b_l)_1^q; x) = \left[\prod_{j=1}^p \frac{\Gamma(b_{q-p+j})}{\Gamma(a_j)}\right] I_{1,p}^{(a_k-1),(b_{q-p+k}-a_k)}\{{}_0F_{q-p}((b_l)_1^{q-p}; x)\}. \tag{3.7}$$

This intermediate differintegral relation combined with (3.5) gives the following particular form of proposition 3.1 in the case $p < q$.

Theorem 3.4. Each ${}_pF_q$-function, $p < q$ is a generalized q-tuple fractional (differ) integral of $\cos_{q-p+1}(x)$, namely:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; -x) = A I_{1,q}^{(\gamma_k),(\delta_k)}\{\cos_{q-p+1}((q-p+1)x^{1/(q-p+1)})\} \tag{3.8}$$

with $A = \sqrt{q-p+1/(2\pi)^{q-p}}[\prod_{j=1}^{q}\Gamma(b_j)/\prod_{i=1}^{p}\Gamma(a_i)]$ and parameters γ_k, δ_k:

$$\gamma_k = \begin{cases} \dfrac{k}{q-p+1} - 1 \\ a_{k-q+p} - 1 \end{cases} \qquad \delta_k = \begin{cases} b_k - \dfrac{k}{q-p+1} & \text{for } k = 1, \ldots, q-p \\ b_k - a_{k-q+p} & \text{for } k = q-p+1, \ldots, q. \end{cases} \tag{3.9}$$

If the conditions

$$ b_k > \frac{k}{q-p+1},\ k = 1, \ldots, q-p \qquad b_{q-p+k} > a_k > 0,\ k = 1, \ldots, p \tag{3.10}$$

are satisfied, then relation (3.8) gives a Poisson-type integral representation:

$$\begin{aligned} {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; -x) &= A \int_0^1 G_{q,q}^{q,0}\left[\sigma \left| \begin{matrix} (b_k)_{k=1}^{q} \\ (k/(q-p+1))_{k=1}^{q-p}, (a_{k-q+p})_{k=q-p+1}^{q} \end{matrix}\right.\right] \\ &\quad \times \sigma^{-1} \cos_{q-p+1}[(q-p+1)(x\sigma)^{1/(q-p+1)}]\, \mathrm{d}\sigma \\ &= A \int_0^1 \underset{(q)}{\cdots} \int_0^1 \prod_{k=1}^{q-p}\left[\frac{(1-t_k)^{b_k-(k/(q-p+1))-1}}{\Gamma(b_k-(k/(q-p+1)))} t_k^{(k/(q-p+1))-1}\right] \\ &\quad \times \prod_{k=q-p+1}^{q}\left[\frac{(1-t_k)^{b_k-a_{k-q+p}-1}}{\Gamma(b_k-a_{k-q+p})} t_k^{a_{k-q+p}-1}\right] \\ &\quad \times \cos_{q-p+1}[(q-p+1)(xt_1\ldots t_q)^{1/(q-p+1)}]\, \mathrm{d}t_1 \ldots \mathrm{d}t_q. \end{aligned} \tag{3.11}$$

If (3.10) are not true at least for one of the indices k, then $I_{1,q}^{(\gamma_k),(\delta_k)}$ in (3.8) is considered as a generalized fractional derivative.

Relation (3.8), including integral representations (3.5) and (3.11) and their differintegral analogues, expresses the ${}_pF_q$-functions, $p < q$, as fractional differintegrals (integrals or derivatives) of the generalized cosine function. This fact generalizes the well known representations like (3.4) and (3.6) of the Bessel function via the cosine and thus, suggests the name *Bessel type GHFs* for the functions ${}_pF_q$ with $p < q$. The Bessel function itself is the simplest special function of this *class*.

Second case: $p = q$

Applying relation (3.1) to a function ${}_pF_p$ consequently $(p-1)$ times, we reach a ${}_1F_1$-function which on its side is representable as an E–K operator of the elementary function $x^\alpha \exp x$:

$$\frac{\Gamma(a)}{\Gamma(b)}\, {}_1F_1(a; b; x) = x^{1-a} I_1^{0,b-a}\{x^{a-1}\,\mathrm{e}^x\} = \begin{cases} I_1^{a-1,b-a}\{\mathrm{e}^x\} & \text{if } b > a \\ D_1^{b-1,a-b}\{\mathrm{e}^x\} & \text{if } b < a. \end{cases} \tag{3.12}$$

Thus, combining (3.1) and (3.12), by p steps we obtain the form of proposition 3.1 in this case.

Theorem 3.5. If $p = q$, each GHF ${}_pF_p(x)$ is an p-tuple fractional integral or derivative of the elementary function $\{x^{a_1-1}\mathrm{e}^x\}$, namely

$$ {}_pF_p(a_1,\ldots,a_p;b_1,\ldots,b_p;x) = \Gamma' x^{1-a_1} I_{1,p}^{(\gamma_k),(\delta_k)}\{x^{a_1-1}\,\mathrm{e}^x\} \tag{3.13}$$

where $\gamma_k = a_k - a_1$, $\delta_k = b_k - a_k$, $k = 1,\ldots,p$ and $\Gamma' = \prod_{j=1}^{p}[\Gamma(b_j)/\Gamma(a_j)]$. If

$$ b_k > a_k > 0 \qquad k = 1,\ldots,p \tag{3.14}$$

this relation yields the following integral representation:

$$ \begin{aligned} {}_pF_p(a_1,\ldots,a_p;b_1,\ldots,b_q;x) &= \Gamma' \int_0^1 G_{p,p}^{p,0}\left[\sigma \left| \begin{matrix} (b_k)_1^p \\ (a_k)_1^p \end{matrix}\right.\right] \sigma^{-1}\exp(x\sigma)\,\mathrm{d}\sigma \\ &= \Gamma' \int_0^1 \underset{(p)}{\cdots} \int_0^1 \prod_{k=1}^{p}\left[\frac{(1-t_k)^{b_k-a_k-1} t_k^{a_k-1}}{\Gamma(b_k-a_k)}\right] \exp(xt_1\ldots t_p)\,\mathrm{d}t_1\ldots\mathrm{d}t_p. \end{aligned} \tag{3.15}$$

The above theorem justifies separating all the GHFs ${}_pF_p$ into a *class of GHFs of confluent type*, involving the confluent hypergeometric function ${}_1F_1(a;b;x) = \Phi(a;b;x)$ as a simplest case.

For parameters not satisfying (3.14), relation (3.13) gives differintegral expressions. For example, we can introduce 'spherical' GHFs of confluent type (by analogy with (3.6)) representable by pure differential operators of $\exp x$.

Corollary 3.6. Let all the differences $a_k - b_k = \eta_k$, $k = 1,\ldots,p$ be non-negative integers. Then, the (differ) integral operator in (3.13) turns into a differential operator D_η of integer order $\eta = \eta_1 + \cdots + \eta_k \geqslant 0$ and of form (2.9), namely

$$ \begin{aligned} &{}_pF_p(b_1+\eta_1,\ldots,b_p+\eta_p;b_1,\ldots,b_p;x) \\ &\qquad = \left[\prod_{j=1}^{p}\frac{\Gamma(b_j)}{\Gamma(b_j+\eta_j)}\right]\left[\prod_{k=1}^{p}\prod_{j=1}^{\eta_k}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + b_k + j - 1\right)\right]\{\exp x\} \\ &\qquad = Q_p(x)\{\exp x\}. \end{aligned} \tag{3.16}$$

The differential representation (3.16) gives an example of how differential formulae for the 'spherical' GHFs introduced in [11] can be used for their explicit calculation, especially in the case $p = q$ in the form $Q_p(x)\{\exp x\}$, where $Q_p(x)$ is a p-degree polynomial. A special case of (3.16) with $b_k = \eta_k = 1$, $k = 1,\ldots,p$ and $Q_p(x) = (\mathrm{d}/\mathrm{d}x)(x(\mathrm{d}/\mathrm{d}x))^p$ can be seen in [14, p 593].

Third case: $p = q + 1$

The generalized hypergeometric functions ${}_pF_q$, $p = q + 1$ are said to be GHFs of Gauss type (see [11], ch 4) and are considered for $|x| < 1$. In this case the starting specific result is the representation of the Gauss hypergeometric function

$$ \frac{\Gamma(a_1)}{\Gamma(b_1)}\,{}_2F_1(a_1,a_2;b_1;x) = I_1^{a_2-1,b_1-a_2}\{(1-x)^{-a_1}\} = x^{1-a_2} I_1^{0,b_1-a_2}\{x^{a_2-1}(1-x)^{-a_1}\} \tag{3.17}$$

for $b_1 > a_2 > 0$, or with an E–K fractional derivative if $a_1 > b_1 > 0$.

Since by $(q-1)$ steps of (3.1) a ${}_pF_q$-function reduces to a ${}_2F_1$-function and the composition of fractional differintegrals in (3.1) and (3.17) gives a q-tuple integral or derivative, we obtain the third form of proposition 3.1.

Theorem 3.7. In the unit disk $|x| < 1$ the GHFs of Gauss type ${}_pF_q$, $p = q + 1$ are q-tuple generalized fractional differintegrals of elementary functions of the form $x^{\alpha}(1-x)^{\beta}$, namely

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x) = \Gamma'' x^{1-a_2} I_{1,q}^{(a_{k+1}-1)_1^q, (b_k - a_{k+1})_1^q} \{x^{a_2-1}(1-x)^{-a_1}\} \tag{3.18} $$

with $\Gamma'' = \prod_{j=1}^{q}[\Gamma(b_j)/\Gamma(a_{j+1})]$. This means that for

$$ b_k > a_k > 0 \qquad k = 1, \ldots, m \tag{3.19} $$

the following Poisson-type integral representations hold:

$$ \begin{aligned} {}_{q+1}F_q(\pm x) &= \Gamma'' \int_0^1 G_{q,q}^{q,0}\left[\sigma \middle| \begin{matrix} (b_k) \\ (a_{k+1}) \end{matrix}\right] \sigma^{-1}(1 \mp x\sigma)^{-a_1}\, \mathrm{d}\sigma = \left[\prod_{j=1}^{q} \frac{\Gamma(b_j)}{\Gamma(a_{j+1})\Gamma(b_j - a_{j+1})}\right] \\ &\quad \times \int_0^1 \underset{(q)}{\cdots} \int_0^1 \prod_{j=1}^{q} [(1-t_k)^{b_k - a_{k+1} - 1} t_k^{a_{k+1}-1}] \times (1 \mp x t_1 \ldots t_q)^{-a_1}\, \mathrm{d}t_1 \ldots \mathrm{d}t_q. \end{aligned} \tag{3.20} $$

The repeated integral form of (3.20) can also be found in [14, p 438].

Corollary 3.8. For $q = 1$, representation (3.20) coincides with (3.17), and can be written as the known Euler formula for the Gauss function ([5, vol 1]):

$$ {}_2F_1(a_1, a_2; b_1; x) = \frac{\Gamma(b_1)}{\Gamma(a_2)\Gamma(b_1 - a_2)} \int_0^1 \frac{(1-t)^{b_1 - a_2 - 1} t^{a_2 - 1}}{(1 - xt)^{a_1}}\, \mathrm{d}t \tag{3.21} $$

valid in $|x| < 1$.

This formula *proposes a way for an analytical continuation of* ${}_2F_1(x)$ outside the unit disk to the domain $|\arg(1 - x)| < \pi$, where the right-hand side of (3.21) represents an analytical function of x. For the same reasons, *formulae (3.20) can serve as analytical extensions of the GHFs* ${}_{q+1}F_q(x), q \geqslant 1$ *outside* $|x| < 1$.

The case with parameters not satisfying condition (3.19) yields generalized fractional derivatives in (3.18) and also provides useful corollaries. By analogy with the previous two cases, we introduce the notion of spherical GHFs of Gauss type when all the differences $a_k - b_k = \eta_k$, $k = 1, \ldots, q$, are non-negative integers and ${}_{q+1}F_q(x)$ is representable by a purely differential operator of a function $(1-x)^{\beta}$ (a special case can be seen in [14, p 572]).

Another interesting case concerns the so-called hypergeometric polynomials

$$ {}_{p+1}F_q(-n, a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^{n} \frac{(-n)_k (a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} \tag{3.22} $$

when $p = q$. By taking $a_{q+1} = -n$, $n \geqslant 0$-integer and $a_k > b_k > 0$, $k = 1, \ldots, q$, the fractional derivative form of (3.18) turns into the Rodrigues-type formula

$$ \begin{aligned} &\left[\prod_{j=1}^{q} \frac{\Gamma(a_j)}{\Gamma(b_j)}\right] {}_{p+1}F_q(-n, a_1, \ldots, a_q; b_1, \ldots, b_q; x) \\ &\quad = D_{1,q}^{(b_k-1),(a_k-b_k)}\{(1-x)^n\} = x^{1-a_q} D_{1,q}^{(b_k-a_q),(a_k-b_k)}\{x^{a_1-1}(1-x)^n\} \\ &\quad = x^{1-b_q} D^{a_q-b_q} x^{a_p-b_{q-1}} D^{a_{p-1}-b_{q-1}} \ldots x^{a_3-b_2} D^{a_2-b_2} x^{a_2-b_1} D^{a_1-b_1}\{x^{a_1-1}(1-x)^n\}. \end{aligned} \tag{3.23} $$

Special cases of (3.23) yield some classical Rodrigues formulae (for more details see [1, 15]). For example, $p = q = 1$ with $a_1 = n + 1$, $b_1 = 1$ and $x \to (1 - x)/2$ yields the Rodrigues formula for the Legendre polynomials,

$$ P_n(x) = (-1)^n\, {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{1-x^n}{2} \frac{1+x^n}{2}\right] $$

$$= \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\{(x^2-1)^n\} \tag{3.24}$$

and $p = q = 2$ with $a_1 = n+1$, $b_1 = 1$, $a_2 = \zeta$, $b_2 = p$ $(\zeta > p > 0)$ gives the Rodrigues formula for the Rice polynomials, viz

$$R_n(x) = {}_3F_2(-n, n+1, \zeta; 1, p; x) = \frac{\Gamma(p)}{n!\Gamma(\zeta)}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^{1-p}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\zeta-p}\right]\{x^n(1-x)^n\}. \tag{3.25}$$

All the results mentioned in this section show that there are, essentially, *three kinds of GHFs* ${}_pF_q$, similar in properties and reducible to the three elementary functions (1.7) and to the three 'initial' (simplest) GHFs ${}_0F_1$, ${}_1F_1$, ${}_2F_1$. Thus, results for them can be obtained by the tools of generalized fractional calculus.

4. New results on the Wright's generalized hypergeometric functions ${}_p\Psi_q$

In the previous section we have reviewed results on the generalized hypergeometric functions using fractional differintegrals with G-functions only. Now we show that the same theory works for the *Wright's GHFs* ${}_p\Psi_q(x)$, $p < q$, $p = q$, $p = q+1$ with operators and representations involving the Fox's H-functions. In this way, we give a solution to the Open problem E.4, stated in [11].

Although the ${}_pF_q$-functions (1.6) encompass almost all the known special functions of mathematical physics, there still exist important examples of generalized hypergeometric functions that could not be included in their scheme and, for arbitrary parameters, are not Meijer's G-functions at all. Happily enough, these cases are representable by means of the Fox's H-functions ([14, 18, 19]). Let us mention the Mittag–Leffler-type functions ([5, vol 3; 18; 11, appendix]):

$$E_\rho(x;\mu) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\mu+(k/\rho))} = x^{\rho(\mu+1)}H^{1,1}_{1,2}\left[-x\left|\begin{matrix}(0,1)\\(0,1),(1-\mu,(1/\rho))\end{matrix}\right.\right]$$
$$\rho > 0, \mu \in \mathbb{C}$$

which have found useful applications as solutions of fractional-order differential and integral equations (see e.g. Tuan and Al-Saqabi [20]) and in many other problems of analysis. They reduce to G-functions only for rational values of the parameter ρ.

Other examples are the ${}_p\Psi_q$-Wright's generalized hypergeometric functions (1.8) and (1.9), generalizing the ${}_pF_q$-functions. Interesting particular cases of them are the so-called Wright's generalized Bessel functions ([21])

$$J^\mu_\nu(x) := {}_0\Psi_1[(\nu+1,\mu);-x] = H^{1,0}_{0,2}[x|(0,1),(-\nu,\mu)] = \sum_{k=0}^{\infty}\frac{(-x)^k}{\Gamma(\nu+k\mu+1)k!}. \tag{4.1}$$

For $\mu = 1$, (4.1) turns into the Bessel functions. The above functions are also misnamed as Bessel–Maitland functions, by the second name Maitland of E M Wright [21].

To develop an approach to the ${}_p\Psi_q$-functions, similar to that for the ${}_pF_q$-functions of section 3, we are to deal with compositions of fractional integrals and derivatives, similar to the E–K operators (2.14) and (2.15) but involving four instead of three arbitrary parameters. The generalized fractional integration operators, suitable for our aims, turn to involve as kernels the Wright's generalized Bessel functions (4.1) instead of the elementary kernels (2.13) of (2.14).

Generalizing the operators and results of Kiryakova [11] and Kalla and Kiryakova [8], in [7] Kalla and Galue have already introduced such operators and their compositions have been shown to be representable by single integrals, closely looking like (2.4). However, their kernel $H^{m,0}_{m,m}$-functions involve *two different groups of positive numbers* $\{\beta_k\}_1^m$ and $\{\lambda_k\}_1^m$ in the bottom and top rows of the parameters. Kalla and Galue have also established properties of these new operators, completely analogous to that of operators (2.4) and (2.10) mentioned in section 2. Therefore, we give here only the basic definitions and omit the details.

Definition 4.1. Let $\beta > 0$, $\lambda > 0$, $\delta \geqslant 0$, γ be real parameters. We define the Wright–Erdélyi–Kober (W–E–K) operators of fractional integration by

$$W^{\gamma,\delta}_{\beta,\lambda} f(x) := I^{\gamma,\delta}_{\beta,\lambda,1} f(x) = \lambda \int_0^1 \sigma^{\lambda(\gamma+1)-1} J^{-\lambda/\beta}_{\gamma+\delta-\lambda(\gamma+1)/\beta}(\sigma^\lambda) f(x\sigma)\,\mathrm{d}\sigma \tag{4.2}$$

where J^μ_ν stands for the Wright's Bessel function (4.1).

If $\lambda = \beta$, (4.2) reduces to the E–K operators (2.14). For $\gamma > -(\mu/\lambda) - 1$, $\delta \geqslant 0$, $\beta \geqslant \lambda > 0$, the W–E–K operators are considered in the classes $\mathcal{H}_\mu(\Omega)$, (2.20) with $\Omega = \{|x| < R, |\arg(1/x)| < (\pi/2)((1/\lambda) - (1/\beta))\}$.

Definition 4.2. With an integer $m \geqslant 1$ and real parameters $\beta_k > 0$, $\lambda_k > 0$, $\delta_k \geqslant 0$, γ_k, $k = 1, \dots, m$, we define the multiple W–E–K fractional integrals by

$$If(x) = I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = \begin{cases} \displaystyle\int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} (\gamma_k+\delta_k+1-(1/\beta_k),(1/\beta_k))_1^m \\ (\gamma_k+1-(1/\lambda_k),(1/\lambda_k))_1^m \end{matrix}\right.\right] f(x\sigma)\,\mathrm{d}\sigma \\ \qquad\qquad \text{if } \displaystyle\sum_{k=1}^m \delta_k > 0 \\ f(x) \quad \text{if } \delta_k = 0, \lambda_k = \beta_k, k = 1, \dots, m. \end{cases} \tag{4.3}$$

It is suggested that

$$\gamma_k > -\frac{\mu}{\lambda_k} - 1 \qquad \delta_k \geqslant 0 \qquad \beta_k \geqslant \lambda_k > 0 \tag{4.4}$$

to ensure that

$$A = \sum_{k=1}^m \frac{1}{\lambda_k} - \sum_{k=1}^m \frac{1}{\beta_k} \geqslant 0$$

and also to preserve the functional space

$$\mathcal{H}_\mu(\Omega) \text{ with } \Omega = \left\{|x| < R, \left|\arg\frac{1}{x}\right| < \frac{\pi}{2}A\right\}.$$

The following two relations, following from [18], (2.2.4) and (2.6.1):

$$\begin{aligned} H^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \gamma+\delta+1-(1/\beta),(1/\beta) \\ \gamma+1-(1/\beta),(1/\beta) \end{matrix}\right.\right] &= \beta G^{1,0}_{1,1}\left[\sigma^\beta \left| \begin{matrix} \gamma+\delta+1-(1/\beta) \\ \gamma+1-(1/\beta) \end{matrix}\right.\right] \\ &= \beta\sigma^{\beta-1}\frac{(1-\sigma^\beta)^{\delta-1}}{\Gamma(\delta)}\sigma^{\beta\gamma} \end{aligned} \tag{4.5}$$

$$H^{1,0}_{1,1}\left[\sigma \left| \begin{matrix} \gamma+\delta+1-(1/\beta),(1/\beta) \\ \gamma+1-(1/\beta),(1/\beta) \end{matrix}\right.\right] = \lambda\sigma^{\lambda(\gamma+1-(1/\lambda))} J^{-\lambda/\beta}_{\gamma+\delta-(\lambda/\beta)(\gamma+1)}(\sigma^\lambda) \tag{4.6}$$

show that if $\beta_k = \lambda_k$, then the new 'multiple W–E–K' fractional integrals coincide with the m-tuple E–K operators (2.4).

Under conditions (4.4), operational rules, similar to (2.22)–(2.30), are satisfied. Symbols $I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}$ with some of the δ_k, $k = 1, \ldots, m$, negative are interpreted as multiple W–E–K fractional derivatives $D^{(\gamma_k+\delta_k),(-\delta_k)}_{(\beta_k),(\lambda_k),m}$, namely

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = \left[\prod_{r=1}^{m}\prod_{j=1}^{\eta_r}\left(\frac{1}{\beta_r} x \frac{\mathrm{d}}{\mathrm{d}x} + \gamma_r + \delta_r + j - \frac{1}{\beta_r}\right)\right] I^{(\gamma_k),(\delta_k+\eta_k)}_{(\beta_k),(\lambda_k),m} f(x). \tag{4.7}$$

By analogy with [11], Kalla and Galue [7] have proved, by mathematical induction, the decomposition relationship

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) = \left[\prod_{k=1}^{m} W^{\gamma_k,\delta_k}_{\beta_k,\lambda_k}\right] f(x) = \left[\prod_{k=1}^{m} I^{\gamma_k,\delta_k}_{\beta_k,\lambda_k,1}\right] f(x) \tag{4.8}$$

justifying the name ‘multiple (m-tuple)’ operators for (4.3).

The basic result established can be stated by the following.

Proposition 4.3. All the Wright’s generalized hypergeometric functions ${}_p\Psi_q(x)$ can be represented as multiple (q-tuple) W–E–K fractional integrals or derivatives of one of the three basic functions

$$\cos_{p-q+1}(x) \qquad \exp x \qquad {}_1\Psi_0(x) = H^{1,1}_{1,1}(-x) \tag{4.9}$$

depending on whether $p < q,\ p = q,\ p = q + 1$.

Naturally, functions (4.9) generalize the three basic functions (1.7), corresponding to the case of ${}_pF_q$-functions and generalized fractional differintegrals involving the simpler G-functions.

The following lemma is the basis of the results in this section.

Lemma 4.4. Each ${}_{p+1}\Psi_{q+1}$-function is representable as a W–E–K fractional integral or derivative of a ${}_p\Psi_q$-function, namely

$$\begin{aligned} {}_{p+1}\Psi_{q+1}&\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p), (\alpha_{p+1}, A_{p+1})\\ (\beta_1, B_1), \ldots, (\beta_q, B_q), (\beta_{q+1}, B_{q+1})\end{matrix}\middle| x\right] \\ &= I^{\alpha_{p+1}-1,\beta_{q+1}-\alpha_{p+1}}_{1/B_{q+1},1/A_{p+1},1}\ {}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix}\middle| x\right] \end{aligned} \tag{4.10}$$

for $|x| < \infty$ ($|x| < 1$ if $p = q + 1$).

The proof of the above relation, as well as the proofs of the particular statements below follow easily by representing all the functions as H-functions and by application of the key formula for an integral of a product of two H-functions ([14, 18; 11, appendix]):

$$\begin{aligned} &\int_0^\infty x^{\alpha-1} H^{s,t}_{u,v}\left[\sigma x\middle|\begin{matrix}(c_i, C_i)_1^u\\ (d_l, D_l)_1^v\end{matrix}\right] H^{m,n}_{p,q}\left[\omega x^r\middle|\begin{matrix}(a_j, A_j)_1^p\\ (b_k, B_k)_1^q\end{matrix}\right] \mathrm{d}x \\ &\quad= \sigma^{-\alpha} H^{m+t,n+s}_{p+v,q+u}\left[\frac{\omega}{\sigma^r}\middle|\begin{matrix}(a_j, A_j)_1^n, (1-d_l-\alpha D_l, rD_l)_1^v, (a_j, A_j)_{n+1}^p\\ (b_k, B_k)_1^m, (1-c_i-\alpha C_i, rC_i)_1^u, (b_k, B_k)_{m+1}^q\end{matrix}\right]. \end{aligned} \tag{4.11}$$

The cases $p < q,\ p = q,\ p = q + 1$ are considered separately, as in section 3.

First case: $p < q$

Definition 4.5. By analogy with the hyper-Bessel functions (3.2), we call the GHFs

$$
\begin{aligned}
&{}_0\Psi_m\left[\begin{array}{c} \\ (\beta_1, B_1), \ldots, (\beta_m, B_m)\end{array}\middle| -x\right] \\
&\qquad = H^{1,0}_{0,m+1}\left[x\middle|\begin{array}{c} \\ (0,1), (1-\beta_1, B_1), \ldots, (1-\beta_m, B_m)\end{array}\right] \\
&\qquad = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta_1 + kB_1)\ldots(\beta_m + kB_m).k!}
\end{aligned} \tag{4.12}
$$

Wright's hyper-Bessel functions. The notation, corresponding to (3.4), (3.2) and (4.1) should be $J^{B_1,\ldots,B_m}_{\beta_1-1,\ldots,\beta_m-1}(-x)$.

The result, analogous to that provided by the Poisson–Dimovski transformation, now takes the following form: *each Wright's hyper-Bessel function* ${}_0\Psi_{q-p}$ *can be represented by means of a Poisson-type integral of the* $\cos_{p-q+1}$*-function, namely*

$$
\begin{aligned}
{}_0\Psi_m\left[\begin{array}{c} \\ (\beta_1, B_1), \ldots, (\beta_{q-p}, \beta_{q-p})\end{array}\middle| -x\right] &= I^{((k/(q-p+1))-1),(\beta_k-(k/(q-p+1))}_{(1/B_k),(1),q-p} \\
&\quad\times\{\cos_{q-p+1}((q-p+1)x^{1/(q-p+1)})\}.
\end{aligned} \tag{4.13}
$$

Thus, combining the p-times application of (4.10), the composition/decomposition property (4.8) and the generalized Poisson integral representation (4.13), we obtain the following.

Theorem 4.6. Each ${}_p\Psi_q$-function, $p < q$, is a generalized q-tuple W–E–K fractional (differ) integral of $\cos_{q-p+1}(x)$, namely

$$
{}_p\Psi_q\left[\begin{array}{c}(\alpha_1, A_1), \ldots, (\alpha_p, A_p) \\ (\beta_1, B_1), \ldots, (\beta_q, B_q)\end{array}\middle| -x\right] = I^{(\gamma_k),(\delta_k)}_{(1/B_k),(\lambda_k),q}\{\cos_{q-p+1}((q-p+1)x^{1/(q-p+1)})\} \tag{4.14}
$$

with parameters

$$
\begin{aligned}
&\gamma_k = \begin{cases} \dfrac{k}{q-p+1} - 1 \\ \alpha_{k-q+p} - 1 \end{cases} \qquad \delta_k = \begin{cases} \beta_k - \dfrac{k}{q-p+1} \\ \beta_k - \alpha_{k-q+p} \end{cases} \\
&\lambda_k = \begin{cases} 1 & k = 1, \ldots, q-p \\ \dfrac{1}{A_{k-q+p}} & k = q-p+1, \ldots, q. \end{cases}
\end{aligned} \tag{4.15}
$$

If the condition

$$
\begin{aligned}
&\beta_k > \frac{k}{q-p+1}, k = 1, \ldots, q-p \qquad \beta_k > a_{k-q+p} > 0,\ k = q-p+1, \ldots, q \\
&B_k \geqslant 1, \qquad\qquad k = 1, \ldots, q-p \qquad B_k \geqslant A_{k-q+p},\ k = q-p+1, \ldots, q
\end{aligned} \tag{4.16}
$$

are satisfied, then relation (4.14) gives a Poisson-type integral representation; otherwise the operator on the right-hand side should be interpreted as a multiple W–E–R derivative (4.7) and (4.14) turns into a new Rodrigues-type formula.

Theorem 4.6 suggests the name Bessel-type Wright's GHFs for the functions ${}_p\Psi_q$ with $p < q$. The Bessel function and the Wright's Bessel function are the simplest special functions of this class.

Second case: $p = q$

Applying relation (4.10) to a function ${}_p\Psi_p$ consequently $(p-1)$ times, one gets to a ${}_1\Psi_1$-function which, on the other hand, is representable as a W–E–K operator of $\exp x$:

$$ {}_1\Psi_1\left[\begin{matrix}(\alpha_1, A_1)\\(\beta_1, B_1)\end{matrix}\middle| x\right] = I_{1/B_1,1/A_1,1}\{\exp x\}. \tag{4.17}$$

Combining (4.10) and (4.17), we obtain the form of proposition 4.3 in this case.

Theorem 4.7. If $p = q$, each GHF ${}_p\Psi_p(x)$ is an p-tuple W–E–K fractional integral or derivative of the exponent function, namely

$$ {}_p\Psi_p\left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p)\\(\beta_1, B_1), \ldots, (\beta_p, B_p)\end{matrix}\middle| x\right] = I_{(1/B_k),(1/A_k),p}\{\exp x\} \tag{4.18}$$

provided

$$\beta_k > \alpha_k > 0 \qquad \text{and} \qquad B_k \geqslant A_k > 0 \qquad k = 1, \ldots, p. \tag{4.19}$$

If for some indices k the ordinance between α_k and β_k is not as in (4.19), representation (4.18) turns into differintegral one.

The above result justifies separating all the GHFs ${}_p\Psi_p$ into a class of Wright's GHFs of confluent type, involving the confluent hypergeometric function ${}_1F_1(a; b; x) = \Phi(a; b; x)$ as a simplest case.

Third case: $p = q + 1$

We call the ${}_{q+1}\Psi_q$-functions Wright's GHFs of Gauss type and consider them for $|x| < 1$. The q-times application of (4.10) gives the following proposition.

Theorem 4.8. In the unit disk $|x| < 1$ each Wright's GHF of Gauss type ${}_p\Psi_q$, $p = q + 1$, is a q-tuple W–E–K fractional differintegral of the $H^{1,1}_{1,1}$-function, generalizing the binomial series from theorem 3.7, namely

$$\begin{aligned} {}_{q+1}\Psi_q\left[\begin{matrix}(\alpha_0, A_0), (\alpha_1, A_1), \ldots, (\alpha_q, A_{q1})\\(\beta_1, B_1), \ldots, (\beta_q, B_q)\end{matrix}\middle| x\right] &= I_{(1/B_k),(1/A_k),q}\left\{{}_1\Psi_0\left[\begin{matrix}(\alpha_0, A_0)\\ \end{matrix}\middle| x\right]\right\} \\ &= I_{(1/B_k),(1/A_k),q}\left\{H^{1,1}_{1,1}\left[-x\middle|\begin{matrix}(1-\alpha_0, A_0)\\(0,1)\end{matrix}\right]\right\} \end{aligned} \tag{4.20}$$

under the same conditions (4.19).

When (4.20) is an integral, it generalizes the Euler integral formula (3.21) for the Gauss function. In particular, for $A_0 = 1$ the basic function in the case $p = q + 1$ reduces to the binomial series

$$ {}_1\Psi_0\left[\begin{matrix}(\alpha_0, 1)\\ \end{matrix}\middle| x\right] = H^{1,1}_{1,1}\left[-x\middle|\begin{matrix}(1-\alpha_0, 1)\\(0,1)\end{matrix}\right] = G^{1,1}_{1,1}\left[-x\middle|\begin{matrix}1-\alpha_0\\0\end{matrix}\right] = (1-x)^{-\alpha_0}. \tag{4.21}$$

The technical details, concerning the results of section 4, together with some interesting particular cases and applications will be given in a separate paper.

The surveyed approach allows, by using the tools of the generalized fractional calculus, the transfer of the known results for the basic functions (1.7) and (4.9) to the complicated generalized hypergeometric functions ${}_pF_q(x)$ and ${}_p\Psi_q(x)$, by separation of the special functions into three classes of functions with similar properties.

In the *physical problems, where these special functions occur*, sometimes physicists and engineers need to dispose with some simple but sufficiently accurate 'approximations' of them. Since, in a sense, the E-K operators and their compositions (i.e. the generalized fractional differintegrals) preserve the asymptotic behaviour and other characteristical properties, one can think for the generalized hypergeometric functions as for objects very close to the well known elementary functions. Namely, all the ${}_pF_q$- and ${}_p\Psi_q$-functions with $p < q$ can be imagined as $\cos_{q-p+1}$-functions, the ${}_pF_p$- and ${}_p\Psi_p$-functions can be thought of as similar to the exponential function and the ${}_{q+1}F_q$-, ${}_{q+1}\Psi_q$-functions—as closest to the functions $x^\alpha(1-x)^\beta$ in the unit disc.

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